The L-functions of Witt coverings

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Abstract. Results on L-functions of Artin-Schreier coverings by Dwork, Bombieri and Adolphson-Sperber are generalized to L-functions of Witt coverings.

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1 Introduction

We shall state our main results after recalling the notion of L-functions of Witt coverings.

Let $\mathbb{F}_q$ be the finite field of characteristic p with q elements, and W_m the ring scheme of Witt vectors of length m over $\mathbb{F}_q$. Let $f \in W_m(\mathbb{F}_q[x_1^{\pm 1}, \cdots, x_n^{\pm 1}])$ with its first coordinate non-constant. Let T^n be the n-dimensional toruse over $\mathbb{F}_q$, and F the Frobenius morphism of W_m. The fibre product over W_m of $W_m \overset{F-1}{\to} W_m$ and $T^n \overset{f}{\to} W_m$ is a $W_m(\mathbb{F}_p)$-covering of T^n, with group action $g(y, x) = (y + g, x)$. The Frobenius element of the Galois group $W_m(\mathbb{F}_p)$ at a closed x of X with degree k is $\mathrm{Tr}_{W_m(\mathbb{F}_{q^k})/W_m(\mathbb{F}_p)}(f(x))$. So the Artin L-function of T^n determined by that $W_m(\mathbb{F}_p)$-covering and a fixed character $\psi : W_m(\mathbb{F}_p) \to \overline{\mathbb{Q}}^\times$ of exact order p^m is

$$L_f(t) = \prod_{x \in |T^n|} (1 - \psi(\mathrm{Tr}_{W_m(\mathbb{F}_{q^k})/W_m(\mathbb{F}_p)}(f(x)))t)^{(-1)^n},$$

where $|T^n|$ is the set of closed points of T^n. By a well known theorem of Deligne [De],

$$L_f(t) = \frac{\prod\limits_{\alpha}(1 - \alpha t)}{\prod\limits_{\beta}(1 - \beta t)},$$

where α and β are algebraic integers such that $q^n \alpha^{-1}$ and $q^n \beta^{-1}$ are also algebraic integers. It implies, as observed by Bombieri [Bo2], $\mathrm{ord}_q(\alpha), \mathrm{ord}_q(\beta) \le n$, where ord_q is the q-order function of $\overline{\mathbb{Q}}_p$ such that $\mathrm{ord}_q(q) = 1$. ($\overline{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$, the field of p-adic numbers.)

By logarithmic differentiation, we get

$$L_f(t) = \exp(\sum_{k=1}^{\infty} S_k(f) \frac{t^k}{k}),$$

where

$$S_k(f) = (-1)^{n-1} \sum_{x \in (\mathbb{F}_{q^k}^{\times})^n} \psi(\mathrm{Tr}_{W_m(\mathbb{F}_{q^k})/W_m(\mathbb{F}_p)}(f(x)))$$

are exponential sums associated to characters of p-power order. To have a look at these exponential sums, we denote by $\lambda_i : A^1 \to W_m$, $i = 0, \cdots, m-1$, the embedding which maps A^1 onto the i-th axis of W_m, and write

$$f = \sum_{i=0}^{m-1} \sum_{u \in I_i} \lambda_i(a_{iu}x^u),$$

where $I_i \subset \mathbb{Z}^n$ and $a_{iu} \in \mathbb{F}_q^{\times}$ are uniquely determined. That decomposition can be obtained by solving the congruences

$$f \equiv \lambda_0(\sum_u a_{0u}x^u)(\mathrm{mod}\ V)$$

$$f - \sum_u \lambda_0(a_{0u}x^u) \equiv \lambda_1(\sum_u a_{1u}x^u)(\mathrm{mod}\ V^2)$$

$$\vdots$$

$$f - \sum_{i=0}^{m-2} \sum_u \lambda_i(a_{0u}x^u) \equiv \lambda_{m-1}(\sum_u a_{(m-1)u}x^u)(\mathrm{mod}\ V^m)$$

successively, where V is the shift operator on W_m.

Let $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$, and ω the Teichmüller lifting from $\overline{\mathbb{F}}_q$ to $\overline{\mathbb{Q}}_p$. We define $\omega(f) = \sum_{i=0}^{m-1} p^i \sum_{u \in I_i} \omega(a_{iu})x^u$. Let $\mathbb{Z}_p$ be the ring of p-adic integers, and μ_l ($l \geq 1$) be the set of l-th roots of unity in $\overline{\mathbb{Q}}_p$. Identifying $W_m(\mathbb{F}_{q^k})$ with $\mathbb{Z}_p[\mu_{q^k-1}]/(p^m)$ under the isomorphism

$$(a_0, \cdots, a_{m-1}) \mapsto \sum_{j=0}^{m-1} \omega(a_i^{p^{-i}})p^i\ (\mathrm{mod}\ p^m),$$

one finds, for $x \in (\mathbb{F}_{q^k}^{\times})^n$, that

$$\psi(\mathrm{Tr}_{W_m(\mathbb{F}_{q^k})/W_m(\mathbb{F}_p)}(f(x))) = \psi(\mathrm{Tr}_{\mathbb{Q}_p[\mu_{q^k-1}]/\mathbb{Q}_p}(\sum_{i=0}^{m-1} \sum_u p^i \omega(a_{iu}^{p^{-i}} x^{p^{-i}u})))$$

$$= \psi(\mathrm{Tr}_{\mathbb{Q}_p[\mu_{q^k-1}]/\mathbb{Q}_p}(\sum_{i=0}^{m-1} \sum_u p^i \omega(a_{iu}x^u))) = \psi(\mathrm{Tr}_{\mathbb{Q}_p[\mu_{q^k-1}]/\mathbb{Q}_p}(\omega(f)(\omega(x)))).$$

Therefore, we have

Lemma 1.1 *For $k = 1, 2, \cdots$, we have*

$$S_k(f) = \sum_{x \in \mu_{q^k-1}^n} \psi(Tr_{\mathbb{Q}_p[\mu_{q^k-1}]/\mathbb{Q}_p}(\omega(f)(x))).$$

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We define the Newton polyhedron $\Delta_\infty(f)$ of f at infinity to be the convex hull in $\mathbb{Q}^n$ of $\{p^{m-i-1}u : 0 \le i \le m-1, u \in I_i\} \cup \{0\}$. Recall that, for a convex polyhedron Δ of dimension n in $\mathbb{Q}^n$ that contains the origin, there is a $\mathbb{R}_{\ge 0}$-linear degree function $u \mapsto \deg(u)$ on $L(\Delta)$, the set of integral points in the cone $\bigcup_{k=1}^{\infty} k\Delta$, such that $\deg(u) = 1$ when u lies on a face of Δ that does not contain the origin. That degree function may take on non-integral values. But there is a positive integer D such that $\deg L(\Delta) \subset D^{-1}\mathbb{Z}$. We denote the least positive integer with this property by $D(\Delta)$. For $k = 0, 1, \cdots$, we denote by $W_\Delta(k)$ the number of points of degree $\frac{k}{D(\Delta)}$ in $L(\Delta)$. We define $P_\Delta(t) = (1 - t^{D(\Delta)})^n \sum_{k=0}^{+\infty} W_\Delta(k)t^k$ for later use. Our first result is an upper bound for the total degree of $L_f(t)$.

Theorem 1.2 *The total degree of $L_f(t)$ is bounded by $\sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{D(n-i+1)} W_\Delta(k)$ with $D = D(\Delta)$ and $\Delta = \Delta_\infty(f)$.*

For $j = 1, \cdots, n$, we write

$$\overline{{}_j f^\tau} = \sum_{i=0}^{m-1} \sum_{p^{m-i-1}u \in \tau} u_j a_{iu}^{p^{m-i-1}} x^{p^{m-i-1}u},$$

where u_j is the j-th coordinate of u. We call f non-degenerate with respect to $\Delta_\infty(f)$ if $\Delta_\infty(f)$ is of dimension n, and for every face τ of $\Delta_\infty(f)$ that does not contain 0, the system $\overline{{}_1 f^\tau} = \cdots = \overline{{}_n f^\tau}$ has no common solution in $(\overline{\mathbb{F}}_q^\times)^n$. Our second result is on L-functions from non-degenerate Witt vectors.

Theorem 1.3 *Suppose that f is non-degenerate with respect to $\Delta := \Delta_\infty(f)$. Then the L-function $L_f(t)$ is a polynomial, and its Newton polygon with respect to ord_q lies above the Hodge polygon of $P_\Delta(t)$ of degree $D(\Delta)$ with the same endpoints. In particular, $L_f(t)$ is of degree $n!\,\mathrm{Vol}(\Delta)$.*

Recall that the Newton polygon of $\prod(1 - \alpha t) \in \overline{\mathbb{Q}}_p[[t]]$ with respect to ord_q is the polygon with vertices at points

$$\left(\sum_{\mathrm{ord}_q(\alpha) \le y} 1, \sum_{\mathrm{ord}_q(\alpha) \le y} \mathrm{ord}_q(\alpha) \right), \ y \in \mathbb{Q}.$$

And the Hodge polygon of $\sum_{k=0}^{+\infty} a_k t^k$ of degree D is the polygon with vertices at the points $(0,0)$ and

$$\left(\sum_{i=0}^{k} a_i, \frac{1}{D} \sum_{i=0}^{k} i a_i \right), \ k = 0, 1, \cdots .$$

Theorem 1.3 was proved by Dwork [Dw] when $m = 1$, and $f(x_1, \cdots, x_n) = x_n h(x_1, \cdots, x_{n-1})$ for some polynomial h with coefficients in $\mathbb{F}_q$. In that case, the L-function $L_f(t)$, by the orthogonality of characters, is related to the zeta function of the hypersurface defined by $h = 0$ in the $(n-1)$-dimensional affine space defined over $\mathbb{F}_q$. It was completely proved by Adolphson-Sperber [AS2] in the case $m = 1$. In the case $n = 1$, the degree of $L_f(t)$ was determined by Kumar-Helleseth-Calderbank [KHC] with applications to coding theory, and by W.-C. W. Li [Li], who read the $p = 2$ version of [KHC].

Our proof of the main results is based on the p-adic method set up by Dwork [Dw, Dw2] and developed by Bombieri [Bo, Bo2], Monsky [Mo], Adolphson-Sperber [AS, AS2], Wan [Wn], and others. The innovation lies in the use of the Artin-Hasse exponential series to produce roots of unity of p-power order.

One can infer the following theorem from Theorem 1.3.

Theorem 1.4 *If f is non-degenerate with respect to $\Delta_\infty(f)$, and the origin lies in the interior of $\Delta_\infty(f)$, then the reciprocal roots of $L_f(t)$ are of absolue value $q^{n/2}$.*

2   The Artin-Hasse exponential series

Let
$$E(t) = \exp(\sum_{i=0}^\infty \frac{t^{p^i}}{p^i}) \in \mathbb{Z}_p[[t]]$$
be the Artin-Hasse exponential series. We shall use it to produce roots of unity of p-power order.

Lemma 2.1 *If l is a positive integer, and π is a root of $\sum_{i=0}^\infty \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$, $E(\pi)$ is a primitive p^l-th root of unity.*

Proof. First, $\exp(p^l \frac{\pi^{p^i}}{p^i})$ exists as $\mathrm{ord}_p(p^l \frac{\pi^{p^i}}{p^i}) \geq \frac{p}{p-1}$. So

$$E(\pi)^{p^l} = E(p^l t)|_{t=\pi} = \prod_{i=0}^\infty \exp(p^l \frac{\pi^{p^i}}{p^i}) = \exp(\sum_{i=0}^\infty p^l \frac{\pi^{p^i}}{p^i}) = \exp(0) = 1.$$

Secondly, as $E(t) \in 1 + t + t^2 \mathbb{Z}_p[[t]]$,

$$E(\pi)^{p^{l-1}} \equiv (1 + \pi)^{p^{l-1}} \equiv 1 + \pi^{p^{l-1}} (\bmod \pi^{p^{l-1}+1}).$$

The lemma is proved.

Lemma 2.2 *Let l be a positive integer. Then the Artin-Hasse exponential series induces a bijection $\pi \mapsto E(\pi)$ from the set of roots of $\sum_{i=0}^\infty \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$ to the set of all primitive p^l-th roots of unity in $\overline{\mathbb{Q}}_p$.*

Proof. The field generated over $\mathbb{Q}_p$ by the set of roots of $\sum_{i=0}^\infty \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$ is precisely $\mathbb{Q}_p(\mu_{p^l})$ since it contains $\mathbb{Q}_p(\mu_{p^l})$ by the preceding lemma, and is of degree no greater than

$p^{l-1}(p-1)$ over $\mathbb{Q}_p$ by Weierstrass' Preparation Theorem. One sees that $E(\tau(\pi)) = \tau(E(\pi))$ if τ is an automorphism $\mathbb{Q}_p(\mu_{p^l})$ over $\mathbb{Q}_p$. So $\pi \mapsto E(\pi)$ maps the set of roots of $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$ onto the set of all primitive p^l-th roots of unity in $\overline{\mathbb{Q}}_p$. It is a bijection as $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} = 0$ has at most $p^{l-1}(p-1)$ roots in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$ by Weierstrass' Preparation Theorem.

Lemma 2.3 *If k is a positive integer, and $x \in \overline{\mathbb{Q}}_p$ satisfies $x^{p^k} = x$, then*

$$E(t)^{x+x^p+\cdots+x^{p^{k-1}}} = E(tx)E(tx^p)\cdots E(tx^{p^{k-1}}).$$

Proof. As $\sum_{j=0}^{k-1} x^{p^j} = \sum_{j=0}^{k-1} x^{p^{j+i}}$, we have

$$E(t)^{x+x^p+\cdots+x^{p^{k-1}}} = \exp(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} \sum_{j=0}^{k-1} x^{p^j}) = \exp(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} \sum_{j=0}^{k-1} x^{p^{j+i}})$$

$$= \exp(\sum_{j=0}^{k-1} \sum_{i=0}^{\infty} \frac{(tx^{p^j})^{p^i}}{p^i}) = E(tx)E(tx^p)\cdots E(tx^{p^{k-1}}).$$

The lemma is proved.

Corollary 2.4 *If π is a root of $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$, and $x \in \overline{\mathbb{Q}}_p$ satisfies $x^{p^k} = x$, then*

$$E(\pi)^{x+x^p+\cdots+x^{p^{k-1}}} = E(\pi x)E(\pi x^p)\cdots E(\pi x^{p^{k-1}}).$$

We now fix an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$. Guaranteed by the above lemma, we may choose, for each $l = 1, \cdots, m$, a unique root π_l of $\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i} = 0$ in $\overline{\mathbb{Q}}_p$ with order $\frac{1}{p^{l-1}(p-1)}$ such that $E(\pi_l) = \psi(1)^{p^{m-l}}$. Let $\Delta = \Delta_\infty(f)$, $D = D(\Delta)$, and π a D-th root of $\pi_m^{p^{m-1}}$ in $\overline{\mathbb{Q}}_p$. For $b \geq 0$, we write

$$L(b) = \{ \sum_{u \in L(\Delta)} a_u x^u : a_u \in \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi], \ \mathrm{ord}_p(a_u) \geq b \deg(u)\}.$$

The Galois group $\mathrm{Gal}(\mathbb{Q}_p[\mu_{q-1}, \pi_m, \pi]/\mathbb{Q}_p)$ acts on $L(b)$ coefficientwise. Define

$$E_f(x) = \prod_{i=0}^{m-1} \prod_{u \in I_i} E(\pi_{m-i}\omega(a_{iu})x^u).$$

Lemma 2.5 *We have $E_f(x) \in L(\frac{1}{p-1})$.*

Proof. Suppose that $0 \leq i \leq m-1$ and $u \in I_i$. We have $p^{m-i-1}u \in \Delta$. So $\deg(p^{m-i-1}u) \leq 1$, and

$$\mathrm{ord}_p(\pi_{m-i}) = \frac{1}{p^{m-i-1}(p-1)} \geq \frac{\deg(p^{m-i-1}u)}{p^{m-i-1}(p-1)} = \frac{\deg(u)}{p-1}.$$

It follows that $\pi_{m-i}\omega(a_{iu})x^u \in L(\frac{1}{p-1})$. Since $E(t) \in \mathbb{Z}_p[[t]]$, we have $E(\pi_{m-i}\omega(a_{iu})x^u) \in L(\frac{1}{p-1})$. The lemma now follows.

Let σ be the Frobenius element of $\mathrm{Gal}(\mathbb{Q}_p[\mu_{q-1}, \pi_m, \pi]/\mathbb{Q}_p)$ fixing π_m and π. The following lemma follows from Corollary 2.4.

Lemma 2.6 *If k is a positive integer, and $x \in \mu^n_{q^k-1}$, then*

$$\psi(Tr_{\mathbb{Q}_p[\mu_{q^k-1}]/\mathbb{Q}_p}(\omega(f)(x))) = \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}).$$

Corollary 2.7 *We have*

$$S_k(f) = (-1)^{n-1} \sum_{x \in \mu^n_{q^k-1}} \prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}), \quad k = 1, 2, \cdots.$$

3   Functions from the Artin-Hasse exponential series

We shall study the growth of the coefficients of $\widehat{_kf}$ $(k = 1, \cdots, n)$, which are defined by

$$d\log \widehat{E}_f(x) = \sum_{k=1}^{n} \widehat{_kf}\frac{dx_k}{x_k}, \quad \widehat{E}_f(x) = \prod_{j=0}^{\infty} E_f^{\sigma^j}(x^{p^j}).$$

Lemma 3.1 *We have*

$$\widehat{_kf} = \sum_{i=0}^{m-1}\sum_{j=0}^{\infty} p^j\gamma_{i,j} \sum_{u \in I_i} u_k\omega(a_{iu}^{p^j})x^{p^ju}, \quad k = 1, \cdots, n,$$

where $\gamma_{i,j} = \sum_{l=0}^{j} \frac{\pi_{m-i}^{p^l}}{p^l}$.

Lemma 3.2 *We have $\pi_l \equiv \pi_m^{p^{m-l}}(\bmod\ \pi_m^{p^{m-l}+1})$.*

Since $E(t) \in 1 + t + t^2\mathbb{Z}_p[[t]]$, we have $E(\pi_l) \equiv 1 + \pi_l(\bmod\ \pi_l^2)$. So we have

$$E(\pi_m)^{p^{m-l}} \equiv (1 + \pi_m)^{p^{m-l}} \equiv 1 + \pi_m^{p^{m-l}}(\bmod\ \pi_m^{p^{m-l}+1}),$$

which, combined with the equality $E(\pi_l) = E(\pi_m)^{p^{m-l}}$, implies that $\pi_l \equiv \pi_m^{p^{m-l}}(\bmod\ \pi_m^{p^{m-l}+1})$.

Corollary 3.3 *We have $\pi_{m-i}^{p^j} \equiv \pi_m^{p^{i+j}}(\bmod\ \pi_m^{p^{i+j}+1})$.*

Lemma 3.4 *If $j \leq m - i - 1$ and $l < j$, we have have*

$$ord_p(\frac{\pi_{m-i}^{p^l}}{p^l}) > ord_p(\frac{\pi_{m-i}^{p^j}}{p^j}).$$

Corollary 3.5 *If $j \leq m - i - 1$, we have $p^j\gamma_{i,j} \equiv \pi_m^{p^{i+j}}(\bmod\ \pi_m^{p^{i+j}+1})$.*

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Corollary 3.6 *Suppowse that $j \leq m - i - 1$. Then $\mathrm{ord}_p(p^j\gamma_{i,j}) > \frac{\deg(p^j u)}{p-1}$ if $\deg(p^{m-i-1}u) \leq 1$, and $\mathrm{ord}_p(p^j\gamma_{i,j} - \pi^{D\deg(p^j u)}) > \frac{\deg(p^j u)}{p-1}$ if $\deg(p^{m-i-1}u) = 1$.*

Lemma 3.7 *If $j \geq m - i$, we have*

$$\mathrm{ord}_p(p^j\gamma_{i,j}) - \frac{\deg(p^j u)}{p-1} \geq p^{j-(m-i)+1} - 1.$$

Proof. Since $\gamma_{i,j} = -\sum_{l=j+1}^{\infty} \frac{\pi_{m-i}^{p^l}}{p^l}$, and $\mathrm{ord}_p(\frac{\pi_{m-i}^{p^l}}{p^l}) \geq \frac{p^{j+1}}{p^{m-i-1}(p-1)} - j + 1$ when $j \geq m - i$ and $l \geq j + 1$, we have we have $\mathrm{ord}_p(p^j\gamma_{i,j}) \geq \frac{p^{j+1}}{p^{m-i-1}(p-1)} - 1$ if $j \geq m - i$. The lemma now follows from the fact that $\deg(p^{m-i-1}u) \leq 1$.

Write

$$B = \{\sum_{u \in L(\Delta)} a_u x^u \in L(\frac{1}{p-1}) : 0 \leq \mathrm{ord}_p(a_u) - \frac{\deg(u)}{p-1} \to +\infty \text{ as } \deg(u) \to \infty\}.$$

Corollary 3.8 *For $k = 1, \cdots, n$, we have $\widehat{_k f} \in B$, and*

$$\widehat{_k f} \equiv \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-1} \sum_{\deg(p^{m-i-1}u)=1} u_k \omega(a_{iu}^{p^j}) \pi^{D\deg(p^j u)} x^{p^j u} \pmod{\pi B}.$$

4 The p-adic trace formula

We shall relate the L-function $L_f(t)$ to the characteristic polynomials of an operator $(p^n F^{-1})^a$ on p-adic spaces.

Since $E_f(x) \in L(\frac{1}{p-1})$ (Lemma 3.1), and $\psi_p : \sum_{u \in L(\Delta)} a_u x^u \mapsto \sum_{u \in L(\Delta)} a_{pu} x^u$ maps $L(b)$ to $L(pb)$, we have the following lemma.

Lemma 4.1 *The map $p^n F^{-1} : g \mapsto \sigma^{-1} \circ \psi_p(E_f(x)g)$ sends $L(\frac{1}{p-1})$ to $L(\frac{p}{p-1})$. In particular, $p^n F^{-1}$ acts on B.*

Note that $p^n F^{-1}$ is σ^{-1}-linear, and $(p^n F^{-1})^a = \psi_p^a \circ \prod_{i=0}^{a-1} E_f^{\sigma^i}(x^{p^i})$ is $\mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$-linear. Write

$$\prod_{i=0}^{ak-1} E_f^{\sigma^i}(x^{p^i}) = \sum_{u \in L(\Delta)} a_u x^u.$$

Then the trace of $(p^n F^{-1})^{ak}$ on B is $\sum_{u \in L(\Delta)} a_{(q^k-1)u}$. And

$$S_k(f) = (-1)^{n-1}(q^k - 1)^n \sum_{u \in L(\Delta)} a_{(q^k-1)u}.$$

So we have the following preliminary trace formula.

Proposition 4.2 *For $k = 1, 2, \cdots$, we have*

$$S_k(f) = -(1 - q^k)^n \, Tr((p^n F^{-1})^{ak}; B).$$

Equivalently,

$$L_f(t) == \prod_{i=0}^n \det(1 - (p^n F^{-1})^a q^i t; B)^{(-1)^i \binom{n}{i}}$$

Let $e_1 = (1, 0, \cdots, 0)$, $\cdots$, $e_n = (0, \cdots, 0, 1)$. For $l = 0, 1, \cdots, n$, we write

$$K_l = \bigoplus_{1 \leq i_1 < \cdots < i_l \leq n} B e_{i_1} \wedge \cdots \wedge e_{i_l}$$

and define

$$p^n F^{-1} : K_l \to K_l, \; g e_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto p^{l+n} F^{-1}(g) e_{i_1} \wedge \cdots \wedge e_{i_l}.$$

Then the preliminary trace formula takes the following form.

Proposition 4.3 *For $k = 1, 2, \cdots$, we have*

$$S_k(f) = \sum_{l=0}^n (-1)^{l+1} \, Tr((p^n F^{-1})^{ak}; K_l).$$

By Corollary 3.9, $\hat{D}_j : g \mapsto (x_j \frac{\partial}{\partial x_j} + \widehat{_j f}) g$, $j = 1, \cdots, n$, operate on B. Obviously, they commute with each other. So, for $l = 1, \cdots, n$,

$$\hat{\partial} : K_l \to K_{l-1}, \; g e_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^l (-1)^{k-1} \hat{D}_{i_k}(g) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \; i_1 < \cdots < i_l$$

are well-defined, and satisfiy $\hat{\partial}^2 = 0$. Thus we get a complex

$$K_n \xrightarrow{\hat{\partial}} K_{n-1} \xrightarrow{\hat{\partial}} \cdots \xrightarrow{\hat{\partial}} K_0.$$

It is easy to check that $p^n F^{-1} \circ \hat{\partial} = \hat{\partial} \circ p^n F^{-1}$. That is, $p^n F^{-1}$ operates on the complex $(K_\bullet, \hat{\partial})$. Therefore we have the following homological trace formula.

Proposition 4.4 *For $k = 1, 2, \cdots$, we have*

$$S_k(f) = \sum_{l=0}^n (-1)^{l+1} \, Tr((p^n F^{-1})^{ak}; H_l(K_\bullet, \hat{\partial})).$$

Equivalently,

$$L_f(t) = \prod_{l=0}^n \det(1 - (p^n F^{-1})^a t; H_l(K_\bullet, \hat{\partial}))^{(-1)^l}.$$

5   The total degree of the L-function

We shall study the Newton polygon of $\det(1 - (p^n F^{-1})^a t; B)$, and then prove Theorem 1.2.

Proposition 5.1 *The Newton polygon of* $\det(1 - (p^n F^{-1})^a t; B)$ *with respect to* ord_q *lies above the Hodge polygon of* $\sum\limits_{i=0}^{+\infty} W_\Delta(k) t^k$ *of degree* D.

Write $E_f(x) = \sum\limits_{u \in L(\Delta)} a_u \pi^{D \deg(u)} x^u$, $a_u \in \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$. Then the matrix of $p^n F^{-1}$ with respect to the orthonormal basis $\{\pi^{D \deg(u)} x^u\}_{u \in L(\Delta)}$, written as a column vector, is

$$A^{\sigma^{-1}} = (a_{pw-u}^{\sigma^{-1}} \pi^{D((p-1)\deg(w)+c(w,u))})_{w,u}, \ c(w,u) = \deg(pw-u) + \deg(u) - p\deg(w) \geq 0.$$

So, the matrix of $(p^n F^{-1})^a$ with respect to that orthonormal basis is $A A^\sigma \cdots A^{\sigma^{a-1}}$. Obviously, the Newton polygon of $\det(1 - At)$ with respect to ord_p lies above the polygon with vertices at points $(0,0)$ and

$$\left(\sum_{i=0}^{k} W_\Delta(i), \sum_{i=0}^{k} W_\Delta(i) \frac{i}{D} \right), \ k = 0, 1, \cdots .$$

It follows that the Newton polygon of $\det(1 - (p^n F^{-1})^a t; B) = \det(1 - A A^\sigma \cdots A^{\sigma^{a-1}} t)$ with respect to ord_q lies above the polygon with vertices at points $(0,0)$ and

$$\left(\sum_{i=0}^{k} W_\Delta(i), \sum_{i=0}^{k} W_\Delta(i) \frac{i}{D} \right), \ k = 0, 1, \cdots .$$

The proposition is proved.

Corollary 5.2 *If* $j \leq n + 1$, *then* $\det(1 - (p^n F^{-1})^a t; B)$ *has at most* $\sum\limits_{k=0}^{Dj} W_\Delta(k)$ *zeros of* q-*order* $\leq j - 1$.

Proof. Define

$$\sum_{k=0}^{+\infty} h_\Delta(k) t^k = (1-t)^n \sum_{k=0}^{+\infty} W_\Delta(k) t^k.$$

Since $\sum\limits_{k=0}^{+\infty} h_\Delta(k) t^k$ is a polynomial of degree $\leq n$ with nonnegative coefficients by a lemma of Kouchnirenko [Ko, Lemma 2.9], and

$$\sum_{k=0}^{jD-i} \binom{n-1+k}{n-1}(k+i) = \binom{n+Dj-i}{n}\left(\frac{n(Dj-i)}{n+1} + i\right) \geq \binom{n+Dj-i}{n} D(j-1),$$

we have

$$\frac{1}{D} \sum_{k=0}^{jD} k W_\Delta(k) = \frac{1}{D} \sum_{k=0}^{jD} k \sum_{i=0}^{k} h_\Delta(i) \binom{n-1+k-i}{n-1}$$

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$$= \frac{1}{D} \sum_{i=0}^{n} h_\Delta(i) \sum_{k=i}^{jD} \binom{n-1+k-i}{n-1} k = \frac{1}{D} \sum_{i=0}^{n} h_\Delta(i) \sum_{k=0}^{jD-i} \binom{n-1+k}{n-1} (k+i)$$

$$\geq (j-1) \sum_{i=0}^{n} h_\Delta(i) \sum_{k=0}^{jD-i} \binom{n-1+k}{n-1} \geq (j-1) \sum_{k=0}^{jD} W_\Delta(k).$$

The corollary now follows from the above inequality by Proposition 5.1.

We now prove Theorem 1.2. Since the reciprocal zeros and reciprocal poles of $L_f(t)$ are of q-order $\leq n$, its total number, by the preliminary trace formula, is bounded by the number of reciprocal zeros of $\prod_{i=0}^{n} \det(1 - (p^n F^{-1})^a q^i t; B)^{\binom{n}{i}}$. By Corollary 5.2, that number is bounded by

$$\sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{D(n-i+1)} W_\Delta(k).$$

Theorem 1.2 is proved.

6   The acyclicity of the p-adic complex

In this section we shall prove the following proposition, which implies the first statement of Theorem 1.3.

Proposition 6.1 *If f non-degenerate with respect to $\Delta_\infty(f)$, then $(K_\bullet, \hat{\partial})$ is acyclic at positive dimensions, and $H_0(K_\bullet, \hat{\partial})$ is a $\mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$-module free of rank $n! \operatorname{Vol}(\Delta_\infty(f))$.*

Write

$$\bar{B} := \mathbb{F}_q[x^{L(\Delta)}] := \{ \sum_{u \in L(\Delta)} a_u x^u : a_u \in \mathbb{F}_q \}.$$

It is a ring with the multiplication rule

$$x^u x^{u'} = \begin{cases} x^{u+u'}, & \text{if } u \text{ and } u' \text{ are cofacial,} \\ 0, & \text{otherwise.} \end{cases}$$

Define

$$B \to \bar{B}, \quad \sum_{u \in L(\Delta)} a_u \pi^{D \deg(u)} x^u \mapsto \sum_{u \in L(\Delta)} \bar{a}_u x^u,$$

where $\bar{a}_u$ is the residue class of a_u modulo the maximal ideal of $\mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$.

Lemma 6.2 *The map $B \to \bar{B}$ is a ring homomorphism. And the sequence*

$$0 \to B \to B \to \bar{B} \to 0$$

is exact.

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For $j = 1, \cdots, n$, we define

$$\bar{D}_j : \bar{B} \to \bar{B}, \ g \mapsto (x_j \frac{\partial}{\partial x_j} + \overline{jf})g,$$

where

$$\overline{jf} = \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-1} \sum_{\deg(p^{m-i-1}u)=1} u_k a_{iu}^{p^j} x^{p^j u}.$$

By Corollary 3.8, we have the following lemma.

Lemma 6.3 *For $j = 1, \cdots, n$, the diagram*

$$\begin{array}{ccc} B & \to & \bar{B} \\ \hat{D}_j \downarrow & & \bar{D}_j \downarrow \\ B & \to & \bar{B} \end{array}$$

is commutative.

For $l = 0, \cdots, n$, we define

$$\bar{K}_l = \bigoplus_{1 \le i_1 < \cdots < i_l \le n} \bar{B} e_{i_1} \wedge \cdots \wedge e_{i_l}.$$

For $l = 1, \cdots, n$, we define

$$\bar{\partial} : \bar{K}_l \to \bar{K}_{l-1}, \ g e_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} \bar{D}_{i_k}(g) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \ i_1 < \cdots < i_l.$$

It is easy to see that the sequence

$$\bar{K}_n \xrightarrow{\bar{\partial}} \bar{K}_{n-1} \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \bar{K}_0$$

is a complex.

Proposition 6.4 *The map $B \to \bar{B}$ induces a morphism of complexes from $(K_\bullet, \hat{\partial})$ to $(\bar{K}_\bullet, \bar{\partial})$. Moreover, the sequence*

$$0 \to (K_\bullet, \hat{\partial}) \to (K_\bullet, \hat{\partial}) \to (\bar{K}_\bullet, \bar{\partial}) \to 0$$

is exact.

Proof. The first statement follows from Lemma 6.3, and the second follows from Lemma 6.2.

By Proposition 6.4, and a lemma of Monsky [Mo, Theorem 8.5], the proof of Proposition 6.1 is reduced to the proof of the following proposition.

Proposition 6.5 *If f is non-degenerate with respect to $\Delta_\infty(f)$, then $(\bar{K}_\bullet, \bar{\partial})$ is acyclic at positive dimensions, and $H_0((\bar{K}_\bullet, \bar{\partial}))$ is a $\mathbb{F}_q$-vector space of dimension $n! \mathrm{Vol}(\Delta_\infty(f))$.*

For $j = 1, \cdots, n$, we define

$$\overline{jf}^0 = \sum_{i=0}^{m-1} \sum_{\deg(p^{m-i-1}u)=1} u_k a_{iu}^{p^{m-i-1}} x^{p^{m-i-1}u}.$$

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For $l = 1, \cdots, n$, we define

$$\bar{\partial}^0 : \bar{K}_l \to \bar{K}_{l-1}, \ ge_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} \overline{i_k f}^0 ge_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \ i_1 < \cdots < i_l.$$

Then

$$\bar{K}_n \xrightarrow{\bar{\partial}^0} \bar{K}_{n-1} \xrightarrow{\bar{\partial}^0} \cdots \xrightarrow{\bar{\partial}^0} \bar{K}_0$$

is a complex. In the next section, we shall prove the following proposition.

Proposition 6.6 *If f is non-degenerate with respect to $\Delta := \Delta_\infty(f)$, then the complex $(\bar{K}_\bullet, \bar{\partial}^0)$ is acyclic at positive dimensions, and the Poincaré series of $H_0((\bar{K}_\bullet, \bar{\partial}^0))$ is $P_\Delta(t)$. In particular, $H_0((\bar{K}_\bullet, \bar{\partial}^0))$ is a $\mathbb{F}_q$-vector space of dimension $n! \, Vol(\Delta)$.*

We now deduce the first statement of Proposition 6.5 from Proposition 6.6. In a given a homology class of positive dimension, we choose one representative ξ of lowest degree. We claim that $\xi = 0$. Otherwise, let ξ^0 be the leading term of ξ. We have $\bar{\partial}^0(\xi^0) = 0$ since it is the leading term of $\bar{\partial}(\xi) = 0$. By the acyclicity of $(\bar{K}_\bullet, \bar{\partial}^0)$, $\xi^0 = \bar{\partial}^0(\eta)$ for some η. The form $\xi - \bar{\partial}(\eta)$ is now of lower degree than ξ, contradicting to our choice of ξ. The proposition is proved.

The second statement of Proposition 6.5 follows the following proposition.

Proposition 6.7 *Let V be a basis of $\bar{K}_0$ modulo $\bar{\partial}^0(\bar{K}_1)$ consisting of homogeneous elements. Then V is also a basis of $\bar{K}_0$ modulo $\bar{\partial}(\bar{K}_1)$.*

Proof. First, we show that $\bar{K}_0$ is generated by V and $\bar{\partial}(\bar{K}_1)$. Otherwise, among elements of $\bar{K}_0$ which are not linear combinations of elements of V and $\bar{\partial}(\bar{K}_1)$, we choose one of lowest degree. We may suppose that it is of form $\bar{\partial}^0(\xi)$. Let ξ^0 be the leading term of ξ. Then $\bar{\partial}^0(\xi) - \bar{\partial}(\xi^0)$ is not a linear combination of elements of V and $\bar{\partial}(\bar{K}_1)$, and is of lower degree than $\bar{\partial}^0(\xi)$. This is a contradiction. Therefore $\bar{K}_0$ is generated by E and $\bar{\partial}(\bar{K}_1)$. It remains to show that $\xi = 0$ whenever ξ belongs to $\bar{\partial}(\bar{K}_1)$ and is a linear combination of elements of V. Otherwise, we may choose one element ζ of lowest degree such that $\xi = \bar{\partial}(\zeta)$. Let ζ^0 be the leading term of ζ. Then $\bar{\partial}^0(\zeta^0)$ is a linear combination of elements of V since it is the leading term of $\bar{\partial}(\zeta)$. So we have $\bar{\partial}^0(\zeta^0) = 0$. By the acyclicity of $(\bar{K}_\bullet, \bar{\partial}^0)$, $\zeta^0 = \bar{\partial}^0(\eta)$ for some η. The form $\zeta - \bar{\partial}(\eta)$ is now of lower degree than ζ, contradicting to our choice of ζ. This completes the proof of the proposition.

7   The complex obtained by reduction

In this section, we shall prove Proposition 6.6. The second statement follows from the first, and the last follows from the second and a lemma of Kouchnirenko [Ko, Lemma 2.9]. So it remains to prove the acyclicity of the complex $(\bar{K}_\bullet, \bar{\partial}^0)$.

Let τ be a face of Δ that does not contain the origin, and $\bar{\tau}$ is the convex hull in $\mathbb{Q}^n$ generated by τ and the origin. For $\alpha_1, \cdots, \alpha_s$ in $\mathbb{F}_q[x^{L(\bar{\tau})}]$, we define $\bar{K}_\bullet(\bar{\tau}, \{\alpha_j\}_{j=1}^s)$ to be the complex

$$\bar{K}_l(\bar{\tau}, \{\alpha_j\}_{j=1}^s) = \bigoplus_{1 \le i_1 < \cdots < i_l \le s} \mathbb{F}_q[x^{L(\bar{\tau})}] e_{i_1} \wedge \cdots \wedge e_{i_l}, \ l = 0, \cdots, s$$

with derivation

$$ge_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} \alpha_{i_k} ge_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \ 1 \le i_1 < \cdots < i_l \le s.$$

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By a proposition of Kouchnirenko [Ko, Proposition 2.6] and the argument of Adolphson-Sperber [AS2, p379], the sequence

$$0 \to \bar{K}_\bullet^\emptyset(f) \to \bigoplus_{\dim \tau = n-1} \bar{K}_\bullet(\bar{\tau}, \{\overline{_jf^\tau}\}_{j=1}^n) \to \cdots \to \bigoplus_{\dim \tau = 0} \bar{K}_\bullet(\bar{\tau}, \{\overline{_jf^\tau}\}_{j=1}^n) \to \bar{K}_\bullet^{-1} \to 0$$

is exact, where τ denotes a face of Δ that does not contain the origin, and

$$\bar{K}_l^{-1} = \begin{cases} \mathbb{F}_q^{\binom{n}{l}}, & \text{if the origin is in the interior of } \Delta \text{ and } 1 \leq l \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

By the exactness of that sequence, the acyclicity of the complex $(\bar{K}_\bullet, \bar{\partial}^0)$ follows from the following lemma.

Lemma 7.1 *Let f be a Witt vector of length m with coefficients in $\mathbb{F}_q[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$. Suppose that f is non-degenerate with respect to $\Delta := \Delta_\infty(f)$ and $\dim \Delta = n$. Let τ be a face of Δ of dimension $s-1$ that does not contain the origin. Then the complex $\bar{K}_\bullet(\bar{\tau}, \{\overline{_jf^\tau}\}_{j=1}^n)$ is acyclic at dimensions $> n-s$.*

Since the sequence

$$0 \to \bar{K}_\bullet(\bar{\tau}, \{\alpha_j\}_{j=1}^{s-1}) \to \bar{K}_\bullet(\bar{\tau}, \{\alpha_j\}_{j=1}^s) \to \bar{K}_\bullet(\bar{\tau}, \{\alpha_j\}_{j=1}^{s-1})[-1] \to 0$$

is exact, Lemma 7.1 follows from the following one.

Lemma 7.2 *Suppose that f is non-degenerate with respect to $\Delta_\infty(f)$. If τ is a face of Δ of dimension $r-1$ that does not contain the origin, then there are $1 \leq i_1 < \cdots < i_r \leq n$ such that the complex $\bar{K}_\bullet(\bar{\tau}, \{\overline{_{i_j}f^\tau}\}_{j=1}^r)$ is acyclic at positive dimensions.*

Proof. There are $\{i_1, \ldots, i_r\} \subset \{1, \cdots, n\}$ and $(\alpha_{kj}) \in \mathbb{Q} \cap \mathbb{Z}_p$ $(1 \leq k \leq n, 1 \leq j \leq r)$ such that $u_k = \alpha_{1k}u_{i_1} + \cdots + \alpha_{rk}u_{i_r}$ for all $u = (u_1, \ldots, u_n) \in L(\bar{\tau})$. Let σ be any face of τ. We have

$$\overline{_jf^\sigma} = \alpha_{1j}\overline{_{i_1}f^\sigma} + \cdots + \alpha_{rj}\overline{_{i_r}f^\sigma}.$$

So $\overline{_{i_1}f^\sigma}, \cdots, \overline{_{i_r}f^\sigma}$ have no common zeros in $(\bar{\mathbb{F}}_q^\times)^n$. By a theorem of Kouchnirenko [Ko, Theorem 6.2], $\overline{_{i_1}f^\tau}, \cdots, \overline{_{i_r}f^\tau}$ generate in $\mathbb{F}_q[x^{L(\bar{\tau})}]$ an ideal of finite codimension. Note that $\mathbb{F}_q[x^{L(\bar{\tau})}]$ is Cohen-Macaulay by a theorem of Hochster [Ho, Theorem 1]. The complex $\bar{K}_\bullet(\bar{\tau}, \{\overline{_{i_j}f^\tau}\}_{j=1}^r)$ is acyclic at positive dimensions by a theorem of Serre [Se, Theorem 3, Chapter IV]. The lemma is proved.

8 The Newton polygon of the L-function

In this section we shall prove the second statement of Theorem 1.3. (The last statement follows from the second by a lemma of Kouchnirenko [Ko, Lemma 2.9].) By the argument of Dwork [Dw2, §7], it suffices to prove the following proposition.

Proposition 8.1 *If f is non-degenerate with respect to $\Delta := \Delta_\infty(f)$, then the Newton polygon of $\det(1 - p^n F^{-1}t; H_0(K_\bullet, \hat{\partial}))$ with respect to ord_p lies above the Hodge polygon of $P_\Delta(t)$ of degree $D(\Delta)$, and their endpoints coincide.*

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Let $\bar{V}$ be a basis of $\bar{K}_0$ modulo $\bar{\partial}^0(K_1)$ consisting of homogeneous elements. By Proposition 6.7, it is also a basis of $\bar{K}_0$ modulo $\bar{\partial}(K_1)$. Define

$$V = \{\sum \omega(a_u)x^u : \sum a_u x^u \in \bar{V}\}.$$

It is a basis of B modulo $\sum_{k=1}^{n} \hat{D}_k B$. For real numbers $b > \frac{1}{p-1}$ and c, we write

$$L(b,c) = \{\sum_{u \in L(\Delta)} a_u x^u : a_u \in \mathbb{Q}_p[\mu_{q-1}, \pi_m, \pi], \ \mathrm{ord}_p(a_u) \geq b \deg(u) + c\}.$$

It is compact with respect to the topology of coefficientwise convergence. Let $V(b,c)$ be the subset of elements of $L(b,c)$ which are finite linear combinations of elements of V. In the next section we shall prove the following proposition.

Proposition 8.2 *If $\frac{1}{p-1} < b < \frac{p}{p-1}$, then*

$$L(b,c) = V(b,c) + \sum_{k=1}^{n} \hat{D}_k L(b, c + b - \frac{1}{p-1}).$$

We now prove the first statement of Proposition 8.1. For each $\xi \in V$, we write

$$p^n F^{-1}(\pi^{D \deg(\xi)}\xi) \equiv \sum_{\eta \in V} c_{\eta,\xi} \pi^{D \deg(\eta)} \eta \ (\mathrm{mod} \ \sum_{k=1}^{n} \hat{D}_k B), \ c_{\eta,\xi} \in \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$$

By Lemma 4.1, $p^n F^{-1}(\pi^{D \deg(\xi)}\xi)$ lies in the space $L(\frac{p}{p-1})$. So, by Proposition 8.2, $c_{\eta,\xi}\pi^{D \deg(\eta)}\eta$ lies in every $L(b)$ with $\frac{1}{p-1} < b < \frac{p}{p-1}$. That is, $\mathrm{ord}_p(c_{\eta,\xi}) \geq (b - \frac{1}{p-1})\deg(\eta)$ for every $\frac{1}{p-1} < b < \frac{p}{p-1}$. Thus we have $\mathrm{ord}_p(c_{\eta,\xi}) \geq \deg(\eta)$. Therefore, the Newton polygon of the characteristic polynomial of $(c_{\eta,\xi})$, which is now the Newton polygon of $\det(1 - p^n F^{-1}t; H_0(K_\bullet, \hat{\partial}))$, lies above the Hodge polygon of $P_\Delta(t)$ of degree D. In particular, $\mathrm{ord}_p(\det(c_{\eta,\xi})) \geq \sum_{\xi \in V} \deg(\xi)$.

It remains to show that the Newton polygon of $\det(1 - p^n F^{-1}t; H_0(K_\bullet, \hat{\partial}))$ share the same endpoints with the Hodge polygon of $P_\Delta(t)$ of degree D. Define

$$\phi_p : L(\frac{p}{p-1}) \to L(\frac{1}{p-1}), \quad \sum_{u \in L(\Delta)} a_u x^u \mapsto \sum_{u \in L(\Delta)} a_u x^{pu}.$$

Obviuosly, $p^n F^{-1} \circ (E_f^{-1} \circ \phi_p \circ \sigma) = 1$ on $L(\frac{p}{p-1})$. At the end of this we shall prove the following proposition.

Proposition 8.3 *Modulo $\sum_{k=1}^{n} \hat{D}_k L(\frac{1}{p-1})$, the space $L(\frac{1}{p-1})$ is generated by $\{\pi^{D \deg(\xi)}\xi : \xi \in V\}$.*

So, for each $\xi \in V$, we can find $b_{\eta,\xi} \in \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$

$$E_f^{-1} \circ \phi_p \circ \sigma(\pi^{D \deg(\xi)p}\xi) \equiv \sum_{\eta \in V} b_{\eta,\xi} \pi^{D \deg(\eta)} \eta \ (\mathrm{mod} \ \sum_{k=1}^{n} \hat{D}_k L(\frac{1}{p-1})).$$

It follows that $(c_{\eta,\xi})(b_{\eta,\xi}) = \text{diag}\{\pi^{D\deg(\xi)(p-1)}, \xi \in V\}$. So $\text{ord}_p(\det(c_{\eta,\xi})) \leq \sum_{\eta\in V} \deg(\eta)$. Therefore

$$\text{ord}_p(\det(c_{\eta,\xi})) = \sum_{\eta\in V} \deg(\eta).$$

That is, the Newton polygon of $\det(1 - p^n F^{-1}t; H_0(K_\bullet, \hat{\partial}))$ share the same endpoints with the Hodge polygon of $P_\Delta(t)$ of degree D.

We now prove Proposition 8.3. Let $\xi = \sum_{u\in L(\Delta)} a_u x^u \in L(\frac{p}{p-1})$. For $N = 0, 1, \cdots$, write

$$\xi^{(N)} = \sum_{u\in L(\Delta), \deg(u)\leq N} a_u x^u \in B.$$ As $\{\pi^{D\deg(\eta)}\eta : \eta \in V\}$ is a basis of B modulo $\sum_{k=1}^{n} \hat{D}_k B$, there

are elements $\xi_k^{(N)} \in B$ $(k = 1, \cdots, n)$ such that

$$\xi^{(N)} - \sum_{k=1}^{n} \hat{D}_k \xi_k^{(N)} = \sum_{\eta\in V} a_\eta^{(N)} \pi^{D\deg(\eta)} \eta.$$

As $L(\frac{1}{p-1})$ is compact with respect to the topology of coefficientwise convergence, the sequence $(\{\xi_k^{(N)}\}_{k=1}^{n}, \{a_\eta^{(N)}\}_{\eta\in V})$, $N = 0, 1, \cdots$, has an adherent point $(\{\xi_k\}_{k=1}^{n}, \{a_\eta\}_{\eta\in V})$ in the space $L(b)^n \times (\mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi])^{|V|}$. Therefore we get

$$\xi - \sum_{k=1}^{n} \hat{D}_k \xi_k = \sum_{\eta\in V} a_\eta \pi^{D\deg(\eta)p^{m-1}} \eta.$$

This completes the proof of Proposition 8.3.

9   The space $L(b, c)$

In this section we shall prove Propositions 8.2.

For $k = 1, \cdots, n$, we write

$$\widehat{_k f}^0 = \sum_{i=0}^{m-1} p^{m-i-1}\gamma_{i,m-i-1} \sum_{\deg(p^{m-i-1}u)=1} u_k\omega(a_{iu}^{p^{m-i-1}})x^{p^{m-i-1}u}.$$

For $l = 1, \cdots, n$, we define

$$\hat{\partial}^0 : K_l \to K_{l-1}, \; ge_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l}(-1)^{k-1}\widehat{_{i_k}f}^0 ge_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \; i_1 < \cdots < i_l.$$

It is easy to see that

$$0 \to (K_\bullet, \hat{\partial}^0) \to (K_\bullet, \hat{\partial}^0) \to (\bar{K}_\bullet, \bar{\partial}) \to 0$$

is an exact sequence of complexex. So we have the following lemma.

Lemma 9.1 *Modulo* $\sum_{k=1}^{n} \widehat{_k f}^0 B$, *the space* B *is generated by* $\{\pi^{D\deg(\xi)}\xi : \xi \in V\}$.

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Corollary 9.2 *If $b > \frac{1}{p-1}$, then*

$$L(b,c) = V(b,c) + \sum_{k=1}^{n} \widehat{_kf}^0 L(b, c+b - \frac{1}{p-1}).$$

Proof. Let $\xi \in L(b,c)$, ξ_v ($v \in \deg L(\Delta)$) its homogeneous part of degree v, and k_v the least integer such that $\operatorname{ord}_p(\pi^{k_v}) \geq bv + c$. Then $\pi^{Dv - k_v} \xi_v \in B$. By the above lemma, we may write

$$\pi^{Dv-k_v}\xi_v = \sum_{\eta \in V, \deg(\eta) \leq v} a_\eta^{(v)} \pi^{D \deg(\eta)} \eta + \sum_{i=1}^{n} \widehat{_if}^0 \eta_i^{(v)},$$

where $a_\eta^{(v)} \in \mathbb{Z}_p[\mu_{q-1}, \pi_m, \pi]$, and $\eta_i^{(v)} \in B$ is of degree $\leq v - 1$. It follows that

$$\xi = \sum_{\eta \in V} \eta \pi^{D \deg(\eta)} \sum_{v \geq \deg(\eta)} a_\eta^{(v)} \pi^{k_v - Dv} + \sum_{i=1}^{n} \widehat{_if}^0 \sum_{v \in \deg L(\Delta)} \pi^{k_v - Dv} \eta_i^{(v)}.$$

It is easy to see that the first term on the right-hand side converges to an element in $V(b,c)$, and the inner sum in the second term converges to an element in $L(b, c+b - \frac{1}{p-1})$. The corollary is proved.

For $k = 1, \cdots, n$, we define

$$D_k : B \to B, \ g \mapsto (x_k \frac{\partial}{\partial x_k} + \sum_{i=0}^{m-1} \sum_{j=0}^{m-i-1} p^j \gamma_{i,j} \sum_{u \in I_i} u_k \omega(a_{iu}^{p^j}) x^{p^j u}) g.$$

For $l = 1, \cdots, n$, we define

$$\partial : K_l \to K_{l-1}, \ g e_{i_1} \wedge \cdots \wedge e_{i_l} \mapsto \sum_{k=1}^{l} (-1)^{k-1} D_k(g) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_l}, \ i_1 < \cdots < i_l.$$

Corollary 9.3 *If $b > \frac{1}{p-1}$, then*

$$L(b,c) = V(b,c) + \sum_{k=1}^{n} D_k L(b, c+b - \frac{1}{p-1}).$$

Proof. Note that $\widehat{_kf}^0 - D_k$ maps $L(b,c)$ to $L(b, c - (b - \frac{1}{p-1})e)$ for some constant $e < 1$. Let $\xi \in L(b,c)$. By the previous corollary and induction, we can find a sequence

$$(\eta_0^{(i)}, \cdots, \eta_n^{(i)}) \in V(b, c+i(1-e)(b - \frac{1}{p-1})) \times L(b, c+(i(1-e)+1)(b - \frac{1}{p-1}))^n, \ i = 0, 1, \cdots$$

such that

$$\xi = \eta_0^{(0)} + \sum_{k=1}^{n} \widehat{_kf}^0 \eta_k^{(0)},$$

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and

$$\sum_{k=1}^{n} (\widehat{kf}^0 - D_k)\eta_k^{(i)} = \eta_0^{(i+1)} + \sum_{k=1}^{n} \widehat{kf}^0 \eta_k^{(i+1)}.$$

One sees immediately that $\sum_{i=0}^{\infty} \eta_0^{(i)}$ converges to an element η_0 in $V(b,c)$, and $\sum_{i=0}^{\infty} \eta_k^{(i)}$ converges to an element η_k in $L(b, c + b - \frac{1}{p-1})$. Moreover, we have $\xi = \eta_0 + \sum_{k=1}^{n} D_k \eta_k$. The corollary is proved.

We now prove Proposition 8.2. Note that $D_k - \hat{D}_k \in L(b, p(\frac{p}{p-1} - b) - 1)$ by Lemma 3.7. So it maps $L(b,c)$ to the space $L(b, c + p(\frac{p}{p-1} - b) - 1)$. Let $\xi \in L(b,c)$. By the previous corollary and induction, we can find a sequence

$$(\eta_0^{(i)}, \cdots, \eta_n^{(i)}) \in V(b, c + i(p - (p-1)b)) \times L(b, c + i(p - (p-1)b) + (b - \frac{1}{p-1}))^n, \ \ i = 0, 1, \cdots$$

such that

$$\xi = \eta_0^{(0)} + \sum_{k=1}^{n} D_k \eta_k^{(0)},$$

and

$$\sum_{k=1}^{n} (D_k - \hat{D}_k)\eta_k^{(i)} = \eta_0^{(i+1)} + \sum_{k=1}^{n} D_k \eta_k^{(i+1)}.$$

One sees immediately that $\sum_{i=0}^{\infty} \eta_0^{(i)}$ converges to an element η_0 in $V(b,c)$, and $\sum_{i=0}^{\infty} \eta_k^{(i)}$ converges to an element η_k in $L(b, c + b - \frac{1}{p-1})$. Moreover, we have $\xi = \eta_0 + \sum_{k=1}^{n} D_k \eta_k$. This completes the proof of Proposition 8.2.

10   The weights of the L-function

We shall prove Theorem 1.4.

Let Δ be a convex polyhedron in $\mathbb{Q}^n$ of dimension n that contains the origin, and S_Δ the sum of the volumes of all its $(n-1)$-dimensional faces that contain 0. Write

$$(1 - t^{D(\Delta)})^n \sum_{i=0}^{+\infty} W_\Delta(i)t^i = \sum_{i=0}^{D(\Delta)n} h_\Delta(i)t^i.$$

Lemma 10.1 *We have*

$$\frac{1}{D(\Delta)} \sum_{i=0}^{nD(\Delta)} ih_\Delta(i) = \frac{n}{2} n! \, Vol(\Delta) - \frac{(n-1)!}{2} S_\Delta.$$

In particular, $\frac{1}{D(\Delta)} \sum_{i=0}^{nD(\Delta)} ih_\Delta(i) = \frac{n}{2} n! \, Vol(\Delta)$ if the origin is an interior point of Δ.

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Proof. Note that

$$W_\Delta(i) = \sum_{k=0}^{i/D(\Delta)} h_\Delta(i - D(\Delta)k) \binom{n-1+k}{n-1}.$$

So

$$\sum_{i \leq D(\Delta)x} (x - \frac{i}{D(\Delta)}) W_\Delta(i) = \sum_{j=0}^{nD(\Delta)} h_\Delta(j) \sum_{0 \leq k \leq x - \frac{j}{D(\Delta)}} (x - \frac{j}{D(\Delta)} - k) \binom{n-1+k}{n-1}$$

$$= \sum_{j=0}^{nD(\Delta)} h_\Delta(j) (\frac{x^{n+1}}{(n+1)!} + (\frac{n}{2} - \frac{j}{D(\Delta)}) \frac{x^n}{n!} + O(x^{n-1})).$$

On the other hand, by [AS, (4.12-13)],

$$\sum_{i \leq D(\Delta)x} (x - \frac{i}{D(\Delta)}) W_\Delta(i) = n! \text{Vol}(\Delta) \frac{x^{n+1}}{(n+1)!} + \frac{(n-1)!}{2} S_\Delta \frac{x^n}{n!} + O(x^{n-1}).$$

The lemma now follows.

We now prove Theorem 1.4. Let α_i, $i = 1, \cdots, n! \text{Vol}(\Delta)$, be the eigenvalues of $q^n F^{-1}$ on $H_0(K_\bullet, \hat{\partial})$. By Theorem 1.2 and Lemma 10.1,

$$\text{ord}_q(\prod_{i=1}^{n! \text{Vol}(\Delta)} \alpha_i) = \frac{n}{2} n! \text{Vol}(\Delta).$$

It is known that the eigenvalues α_i are l-adic units when l is a prime different from p. So, by the product formula, we have

$$\prod_{i=1}^{n! \text{Vol}(\Delta)} |\alpha_i| = q^{\frac{n}{2} n! \text{Vol}(\Delta)}.$$

By a theorem of Kedlaya [Ke, Theorem 5.6.2], the Frobenius F on $H_0(K_\bullet, \hat{\partial}) \otimes \mathbb{Q}_p[\mu_{q-1}, \pi_m, \pi]$ is of mixed weight $\geq n$. So $q^n F^{-1}$ on $H_0(K_\bullet, \hat{\partial})$ is of mixed weight $\leq 2n - n \leq n$. That is, $|\alpha_i| \leq q^{n/2}$. It follows that all the eigenvalues α_i must have absolute value $q^{n/2}$. This completes the proof of Theorem 1.4.

11 Applications to other situations

Let J be a subset of $\{1, \cdots, n\}$. For $\{j_1, \cdots, j_s\} \subseteq J$, we write

$$B_{\{j_1, \cdots, j_s\}} = \{\sum_{u \in L(\Delta)} a_u x^u \in B : u_{j_1}, \cdots, u_{j_s} > 0\}.$$

For $l = 0, 1, \cdots, n$, we define

$$K_l(f, J) = \bigoplus_{1 \leq i_1 < \cdots < i_l \leq n} B_{J \setminus \{i_1, \cdots, i_l\}} e_{i_1} \wedge \cdots \wedge e_{i_l}.$$

Then $(K_\bullet(f, J), \hat{\partial})$ is a subcomplex of $(K_\bullet(f, \emptyset), \hat{\partial})$. The latter is the complex $(K_\bullet, \hat{\partial})$ we defined earlier.

Lemma 11.1 *The sequence*

$$0 \to K_\bullet(f, J) \to K_\bullet(f, J \setminus \{j\}) \to K_\bullet(f^{\{j\}}, J \setminus \{j\}) \to 0$$

is exact, where $f^{\{j\}}$ is the Witt vector whose i-th coordinate is the sum of monomials of the i-th coordinate of f not divided by x_j.

We define, for $k = 1, 2, \cdots$,

$$S_k(f, J) = \sum_{x^{q^k} = x, x_{i_1} \cdots x_{i_r} \neq 0} \psi(\mathrm{Tr}_{\mathbb{Q}_p[\mu_{q^k - 1}]/\mathbb{Q}_p}(\omega(f)(x)))$$

if $\{1, \cdots, n\} \setminus J = \{i_1, \cdots, i_r\}$, and $f \in W_m(\mathbb{F}_q[x_1, \cdots, x_n, (x_{i_1} \cdots x_{i_r})^{-1}])$. Here the equation $x^{q^k} = x$ is solved in $(\overline{\mathbb{Q}}_p)^n$. We write

$$L_{f,J}(t) = \exp\left(\sum_{k=1}^\infty S_k(f, J) \frac{t^k}{k}\right).$$

By the above lemma we infer the following trace formula from the earlier one.

Proposition 11.2 *For $k = 1, 2, \cdots$, we have*

$$S_k(f, J) = \sum_{l=0}^n (-1)^{l+1} Tr((p^n F^{-1})^{ak}; H_l(K_\bullet(f, J), \hat{\partial})).$$

Equivalently,

$$L_{f,J}(t) = \prod_{l=0}^n \det(1 - (p^n F^{-1})^a t; H_l(K_\bullet(f, J), \hat{\partial}))^{(-1)^l}.$$

We call f commode with respect to J if $\Delta_\infty(f)$ is commode with respect to J. Recall that a convex polyhedron Δ in $\mathbb{Q}^n$ that contains the origin is commode with respect to J if it lies in $(\prod_{i=1, i \notin J}^n \mathbb{Q}) \times (\prod_{i \in J} \mathbb{Q}_{\geq 0})$ and $\dim(\Delta_C) = n - |C|$ for all subset C of J, where $\Delta_C = \{(u_1, \ldots, u_n) \in \Delta : u_j = 0 \text{ if } j \in C\}$. By Lemma 11.1 and Proposition 11.2, we infer the following proposition from Theorem 1.3.

Proposition 11.3 *If f is commode with respect to J and non-degenerate with respect to $\Delta_\infty(f)$, then $L_{f,J}(t)$ is a polynomial, and its Newton polygon with respect to ord_q lies above the Hodge polygon of*

$$\sum_{C \subset J} (-1)^{|C|} P_{\Delta_C}(t^{\frac{D(\Delta)}{D(\Delta_C)}})$$

with the same endpoints. In particular, $L_{f,J}(t)$ is of degree

$$\sum_{C \subset J} (-1)^{|C|} (n - |C|)! \, Vol(\Delta_C).$$

From Lemma 10.1 we infer the following one.

Lemma 11.4 *Let Δ be a convex polyhedron in $\mathbb{Q}^n$ of dimension n that contains the origin and is commode with respect to J. Let $(V_{\Delta,J}, U_{\Delta,J})$ be the endpoint of the Hodge polygon of*

$$\sum_{C \subset J} (-1)^{|C|} P_{\Delta_C}(t^{\frac{D(\Delta)}{D(\Delta_C)}})$$

other than $(0,0)$. Then

$$U_{\Delta,J} = \frac{n}{2} V_{\Delta,J} + \sum_{l=1}^{|J|+1} (-1)^l \frac{(n-l)!}{2} (\sum_{C \subset J, |C|=l-1} S_{\Delta_C} - l \sum_{C \subset J, |C|=l} Vol(\Delta_C)).$$

In particular, $U_{\Delta,J} = \frac{n}{2} V_{\Delta,J}$ if the origin is an interior point of Δ_J.

By Lemma 10.3 we infer the following proposition from Proposition 10.2.

Proposition 11.5 *If f is commode with respect to J and non-degenerate with respect to $\Delta := \Delta_\infty(f)$, and the origin lies in the interior of Δ_J, then the reciprocal roots of $L_{f,J}(t)$ are of absolue value $q^{n/2}$.*

References

[AS] A. Adolphson and S. Sperber, Newton polyhedra and the degree of the L-function associated to an exponential sum, Invent. Math. 88 (1987), 555-569.

[AS2] A. Adolphson and S. Sperber, Exponential sums and Newton polyhedra: cohomology and estimates, Ann. Math. 130 (1989), 367-406.

[Bo] E. Bombieri, On exponential sums in finite fields, Amer. J. Math. 88 (1966), 71-105.

[Bo2] E. Bombieri, On exponential sums in finite fields II, Invent. Math. 47 (1978), 29-39.

[De] P. Deligne, La conjecture de Weil II, Publ. Math. I.H.E.S. 52 (1980), 137-252.

[Dw] B. Dwork, On the zeta function of a hypersurface, Publ. Math. I.H.E.S. 12 (1962), 5-68.

[Dw2] B. Dwork, On the zeta function of a hypersurface II, Ann. Math. 80 (1964), 227-299.

[Ho] M. Hochster, Rings of invariants of tori, Cohen-Macaulay rings generated by monomials, and polytopes, Ann. Math. 96 (1972), 318-337.

[Ke] K. S. Kedlaya, Fourier tranforms and p-adic Weil II, arXiv: math.NT/0210149.

[Ko] A. G. Kouchnirenko, Polyèdres de Newton et nombres de Milnor, Invent. Math. 32 (1976), 1-31.

[KHC] P. V. Kumar, T. Helleseth and A. R. Calderbank, An upper bound for some exponential sums over Galois rings with applications, IEEE Trans. Inform. Theory 41 (1995), no.2, 456-468.

[Li] W.-C. W. Li, Character sums over p-adic fields, J. Number Theory 74 (1999), no.2, 181-229.

[Mo] P. Monsky, p-Adic analysis and zeta functions, Lectures in Mathematics, Kyoto University, Tokyo.

[Se] J-P. Serre, Algèbre locale. Multiplicites, Lecture Notes in Math. 11, Springer, Berlin-Heidelberg-New York, 1965.

[Wn] D. Wan, Newton polygons of zeta-functions and L-functions, Ann. Math. 137 (1993), 247-293.